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An Interpolation Formula for Harmonic Functions

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1. INTRODUCTION AND RESULTS

Let $u(x, y)$ be a harmonic function in R^2. We call $u(x, y)$ an even function if $u(x, y) = u(-x, -y)$ and an odd function if $u(x, y) = -u(-x, -y)$. In [2], Boas proved the following uniqueness theorem:

THEOREM A. *If $u(x, y)$ is a real-valued entire harmonic function of exponential type less than π and $u(m, 0) = u(m \cos \alpha, m \sin \alpha) = 0$, for $m = 0$, $\pm 1, \pm 2, \ldots$, then $u(x, y) = 0$ unless α is a rational multiple of π.*

In case $\alpha = \pi/2$, the theorem (without "unless...") fails; consider the functions xy and $\sinh x \sin y$. However, since these functions are even, it is possible that it might still hold when u is an odd function. This is in fact the case and we can actually construct the function $u(x, y)$ from its values at the lattice points $(0, n)$ and $(n, 0)$, $n = 0, \pm 1, \ldots$, if $\{u(n, 0)\}$ and $\{u(0, n)\}$ are in l^p. For the case when the lattice points lie on parallel lines, see [1, 2, 4].

THEOREM 1. *Let $u(x, y)$ be a real-valued odd entire harmonic function of exponential type less than π. Let $u(m, 0) = u(0, m) = 0$ for all integers m. Then $u(x, y)$ vanishes identically.*

THEOREM 2. *Let $u(x, y)$ be a real-valued odd entire harmonic function of exponential type less than or equal to π such that the series $\sum_{n=-\infty}^{\infty} | u(0, n)|^p$ and $\sum_{n=-\infty}^{\infty} | u(n, 0)|^p$ are convergent, where $1 \leqslant p < \infty$. Then*

$$u(x, y) = \sum_{n=-\infty}^{\infty} u(n, 0)\, w_n(x, y) + \sum_{n=-\infty}^{\infty} u(0, n)\, w_n(y, x), \tag{1}$$

where

$$w_n(x, y) = \frac{(-1)^n\, n[(x^2 - y^2 - n^2) \cosh \pi y \sin \pi x + 2xy \sinh \pi y \cos \pi x]}{\pi[y^2 + (x - n)^2][y^2 + (x + n)^2]}$$

and the series converge uniformly on every compact subset of R^2.

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COROLLARY 1. *Let $u(x, y)$ be a real-valued harmonic function of exponential type less than π such that $u(0, n) = u(n, 0) = 0$. Then $u(x, y)$ is even.*

COROLLARY 2. *Let $u(x, y)$ be a real-valued odd harmonic function of exponential type less than or equal to π. Then $u(x, 0)$ and $u(0, y)$ are in $L^2(-\infty \;\; \infty)$ if and only if $\{u(n, 0)\}$ and $\{u(0, n)\}$ are in l^2.*

2. PROOFS OF THEOREMS AND CORROLLARIES

To prove Theorem 1, we let $v(x, y)$ be a harmonic function conjugate to u so that $f(x + iy) = u(x, y) + iv(x, y)$ is an odd entire function of exponential type less than π. This is possible by Carathéodory's inequality [2, 3]. We define $F(z) = f(iz) + \overline{f(i\bar{z})}$. As $F(z)$ vanishes at $(n, 0)$ for all integers, $F(z)$ is the zero function by Carlson's theorem. Hence we have

$$\begin{aligned} f(iz) &= -\overline{f(i\bar{z})}, \\ \text{or} \quad f(-z) &= -\overline{f(\bar{z})}. \end{aligned} \tag{2}$$

Similarly, we can conclude that

$$f(z) + \overline{f(\bar{z})} = 0. \tag{3}$$

Now it follows from (2) and (3) that $f(z)$ is even, which implies that $u(x, y)$ is even and hence vanishes identically.

To prove Theorem 2, we observe that

$$w_n(x, y) = -w_n(-x, -y), \tag{4}$$

$$w_n(0, y) = 0, \tag{5}$$

$$w_n(x, 0) = \frac{\sin \pi(x - n)}{2\pi(x - n)} - \frac{\sin \pi(x + n)}{2\pi(x + n)} \tag{6}$$

and

$$w_n(x, y) = O(1/n) \tag{7}$$

uniformly in every compact subset of R^2 as n tends to infinity. Thus, the series in (1) converge uniformly, and by Schwarz's inequality and (7), the rate of convergence is of order $0(1/\sqrt{n})^{q-1}$. An easy way to show that $w_n(x, y)$ is harmonic is from the following equality:

$$w_n(x, y) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \cosh ty (e^{it(x-n)} - e^{-it(x+n)})\, dt. \tag{8}$$

Now, we let

$$w(x, y) = \sum_{n=-\infty}^{\infty} u(n, 0)\, w_n(x, y) + \sum_{n=-\infty}^{\infty} u(0, n)\, w_n(y, x),$$

and $F(z)$ be an entire function of exponential type less than or equal to π such that $\operatorname{Re} F = u(x, y)$. Then, we have

$$u(x, 0) = \frac{1}{2} F(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(x-n)}{x-n} F(n)$$

$$= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(x-n)}{x-n} u(n, 0)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\sin \pi(x-n)}{x-n} [u(n, 0) - u(-n, 0)]$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{\sin \pi(x-n)}{x-n} - \frac{\sin \pi(x+n)}{x+n}\right] u(n, 0).$$

From (5) and (6), we obtain

$$u(x, 0) = \sum_{n=-\infty}^{\infty} u(n, 0)\, w_n(x, 0) + \sum_{n=-\infty}^{\infty} u(0, n)\, w_n(0, y) = w(x, 0).$$

Similarly, we can conclude that

$$u(0, y) = w(0, y).$$

Let $h(z)$ be an odd entire function such that $\operatorname{Re} h = u - w$. We consider $H(z) = h(iz) + \overline{h(i\bar{z})}$. Then $H(iy) = 0$ for all real y. Thus, $h(iz) = \overline{-h(i\bar{z})}$. Similarly, we have $h(z) = \overline{-h(\bar{z})}$, and hence, $u(x, y) - w(x, y) \equiv 0$ as in the proof of Theorem 1.

Corollary 1 follows trivially from Theorem 1 by considering the odd part of u, i.e., $[u(x, y) - u(-x, -y)]/2$.

To prove Corollary 2, we let $f(z)$ be an entire function such that $\operatorname{Re} f = u$ and let $F(z) = f(iz) + \overline{f(i\bar{z})}$. It follows from Paley–Wiener's Theorem (cf. [2]) that

$$F(z) = \int_{-\pi}^{\pi} e^{izt} \phi(t)\, dt$$

for some $\phi \in L^2[-\pi, \pi]$. Hence,

$$\sum_{n=-\infty}^{\infty} u^2(0, n) = \tfrac{1}{4} \sum_{n=-\infty}^{\infty} F^2(n) = \tfrac{1}{4} \int_{-\pi}^{\pi} |\, \phi(t)|^2\, dt < \infty.$$

On the other hand, if $\sum_{n=-\infty}^{\infty} [u^2(0, n) + u^2(n, 0)]$ is convergent, then we have from (5) and (6) that

$$u(x, 0) = 2 \sum_{n=-\infty}^{\infty} u(n, 0) \frac{\sin \pi(x - n)}{\pi(x - n)}.$$

As the sequence $\{[\sin \pi(x - n)]/[\pi(x - n)]\}$ is an orthonormal sequence in $L^2(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} | u(x, 0)|^2 \, dx = 4 \sum_{n=-\infty}^{\infty} [u(n, 0)]^2 < \infty.$$

3. Final Remark

It can be seen from (8) that the construction of $w_n(x, y)$ is motivated by the well-known image method for constructing Green's function for the Laplace operator (cf. [5]). We conjecture that for general $\alpha = q/p$, this method can be extended to give an interpolation formula analogous to (1) for a certain class of nonsymmetric harmonic functions for which a uniqueness theorem holds. Also, the interpolation formula will be a sum of a certain series over some reflected images of (x, y) by the straight lines $y = x \tan k/p \, \pi$, $k = 0, 1,..., p - 1$. Finally, we would like to mention an application of (1). Suppose $u(x, y)$ satisfies the hypothesis of Theorem 2, then the harmonic conjugate v of u can be written as follows:

$$v(x, y) = \sum_{n=-\infty}^{\infty} u(n, 0) \, v_n(x, y) - \sum_{n=-\infty}^{\infty} u(0, n) \, v_n(y, x),$$

where

$$v_n(x, y) = \frac{(-1)^n \, n[(x^2 - y^2 - n^2) \sinh \pi y \cos \pi x - 2xy \cosh \pi y \sin \pi x]}{\pi[(x - n)^2 + y^2][(x + n)^2 + y^2]}.$$

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